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Nonclassical Properties of Orthonormalized Eigenstates of the Operator $(a_q f(N_q))^k$

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In this paper, the completeness of the k orthonormalized eigenstates of the operator $(a_q f(N_q))^k (k \geq 3)$ is proved. We introduce a new kind of higher order squeezing and an antibunching. The properties of the Mth-order squeezing and the antibunching effect of the k states are investigated. The result shows that these states may form a complete Hilbert space, and the Mth order $[M = (m + 1/2)k; m = 0, 1, 2, \ldots]$ squeezing effects exist in all of the k states when k is even. There is the antibunching effect in all of the states.

KEY WORDS: operator $(a_q f(N_q))^k$; orthonormalized eigenstates; completeness; higher order squeezing; antibunching effect.

1. INTRODUCTION

The coherent states introduced by Glauber (1963) are eigenstates of the boson annihilation operator a, and have widespread applications in the field of physics (Ali *et al.*, 2000; Klauder and Skagerstam, 1985; Perelomov, 1986; Zhang *et al.*, 1990a). The even and odd coherent states (Dodonov *et al.*, 1974), which are two orthonormalized eigenstates of the square a^2 of the operator a, play an important role in quantum optics (Hillery, 1987; Xia and Guo, 1989; Bužek *et al.*, 1992). The k orthonormalized eigenstates of the kth power $a^k (k \geq 3)$ of the operator a were constructed by us and applied to quantum optics (Sun *et al.*, 1991, 1992). The notion of coherent states was extended to q-coherent states (Biedenharn, 1989), which are eigenstates of the q-boson annihilation operator a_q. The q-coherent states

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were well studied and applied widely to quantum optics and mathematical physics (Biedenharn, 1989; Bužek, 1991; Chaichian *et al.*, 1990; Gray and Nelson, 1990; Solomon and Katriel, 1990). The even and odd q-coherent states, defined as two orthonormalized eigenstates of the square a_q^2 of the operator a_q, have nonclassical effects (Wang and Kuang, 1992). Moreover, the k orthonormalized eigenstates of the kth power a_q^2 were well investigated by Kuang *et al.* (1993) and applied to quantum optics by us (Wang *et al.*, 1995).

Recently there has been much interest in the study of nonlinear coherent states called f-coherent states (Man'ko *et al.*, 1997), which are eigenstates of the annihilation operator $af(n)$ of f-oscillators, where $f(n)$ is an operator-valued function of the boson number operator n. A class of f-coherent states can be realized physically as the stationary states of the center-of-mass motion of a trapped ion (Matos Filho and Vogel, 1996). The f-coherent states exhibit nonclassical features such as squeezing and self-splitting. Subsequently, the even and odd nonlinear coherent states, which are two orthonormalized eigenstates of the square $(af(n))^2$ of the operator $af(n)$, were constructed and their nonclassical effects were studied (Mancini, 1997; Sivakumar, 1998). Based on this work, the k orthonormalized eigenstates of the kth power $(a_q f(N_q))^k$ $(k \geq 1)$ were construced and their some properties were discussed by Liu (2000). At this stage a natural question arises: whether the k eigenstates could construct a complete Hilbert space, i.e., whether they could be used as a representation. For $k \geq 3$, are they classical or nonclassical? Based on Liu's work (Liu, 2000), in this paper we prove the completeness of the k eigenstates of the operator $(a_q f(N_q))^k$ $(k \geq 3)$. We introduce a new kind of higher order squeezing and an antibunching, and study the nonclassical properties of the k eigenstates, including higher order squeezing and the antibunching effects.

2. COMPLETENESS OF THE *k* ORTHONORMALIZED EIGENSTATES OF $(a_q f(N_q))^k$

The q-boson annihilation operator a_q, creation operator a_q^+, and number operator N_q satisfy the quantum Heisenberg–Weyl algebra:

$$a_q a_q^+ - q a_q^+ a_q = q^{-N_q}, \tag{1}$$

$$[N_q, a_q] = -a_q, \qquad [N_q, a_q^+] = a_q^+, \tag{2}$$

with q is real and positive. The operators a_q, a_q^+, and N_q act in a Hilbert space with the basis $|n\rangle$ $(n = 0, 1, 2, \ldots)$, such that

$$a_q|0\rangle = 0, \qquad |n\rangle = \frac{(a_q^+)^n}{\sqrt{[n]!}}|0\rangle, \tag{3}$$

where the q-factorial $[n]!$ is given by the relation

$$[n]! = [n][n-1]\cdots[1], \qquad [0]! = 1, \tag{4}$$

and the function $[n]$ is defined as:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{5}$$

From above it follows that

$$a_q|n\rangle = \sqrt{[n]}|n-1\rangle, \qquad a_q^+|n\rangle = \sqrt{[n+1]}|n+1\rangle, \qquad N_q|n\rangle = n|n\rangle. \tag{6}$$

Note that $[n]$ is invariant under $q \leftrightarrow 1/q$. For the case of physical relevance, we reasonably choose $0 < q \le 1$. Therefore, it is always true that $[n] \ge n$.

The k orthonormalized eigenstates of the operator $(a_q f(N_q))^k \equiv A^k$ (here and henceforth $k \ge 3$; we do not indicate it in the following) with the same eigenvalue α^k are (Liu, 2000)

$$|\psi_j(\alpha, f)\rangle_k = C_j \sum_{n=0}^{\infty} \frac{\alpha^{kn+j}}{\sqrt{[kn+j]!}\, f(kn+j)!}|kn+j\rangle \equiv |\psi_j\rangle_k \tag{7}$$

where α is a complex number, and $j = 0, 1, 2, \ldots, k-1$. C_j are normalized factors, and f is chosen to be real and nonnegative:

$$f(kn+j)! = f(kn+j) f(kn+j-1) \cdots f(1), \quad f(0)! = 1. \tag{8}$$

It is easy to check that for the same value of k, the k states given by (7) are orthogonal to each other with respect to the subscript j. Let $x = |\alpha|^2$, using the normalized conditions of the states given by (7), we have (Liu, 2000)

$$C_j = \{A_j(x, f)\}^{-1/2} = \left[\sum_{n=0}^{\infty} \frac{x^{kn+j}}{[kn+j]!(f(kn+j)!)^2}\right]^{-1/2} \tag{9}$$

In particular, for $k = 1$, from (7) and (9) we have

$$|\psi_0(\alpha, f)\rangle_1 = \left[\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{[n]!(f(n)!)^2}\right]^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!}\, f(n)!}|n\rangle \equiv |\alpha, f\rangle_q. \tag{10}$$

The states are eigenstates of the operator $a_q f(N_q) = A$. Obviously, this is a natural generalization of the notion of f-coherent states in the q-deformed situation. Therefore, we call, the states $|\alpha, f\rangle_q$ as the $q - f$-coherent states (Liu, 2000).

Now, the question that concerns us is whether the k states given by (7) could construct a complete Hilbert space, i.e., whether they could be used as a representation. In order to construct the completeness formula of the k states, we use the density operator method (Hao, 1993). We define the density operator (i.e., the density matrix) of the state $|kn+j\rangle$ as

$$\rho_j = \sum_{n=0}^{\infty} P(kn+j)|kn+j\rangle\langle kn+j|, \tag{11}$$

where $P(kn+j) = \int P(kn+j,\alpha)\, d^2\alpha$ is the probability distribution of the $(kn+j)$th state $|kn+j\rangle$ appearing in the state $|\psi_j\rangle_k$ in which

$$P(kn+j,\alpha) = |\langle kn+j|\psi_j\rangle_k|^2 = \frac{1}{A_j(|\alpha|^2)} \frac{|\alpha|^{2(kn+j)}}{[kn+j]!(f(kn+j)!)^2}. \tag{12}$$

Thus we have

$$\rho_j^{-1} = \sum_{n=0}^{\infty} P^{-1}(kn+j)|kn+j\rangle\langle kn+j|. \tag{13}$$

Therefore, the completeness formula of the k states given by (7) can be written in the following form:

$$\sum_{j=0}^{k-1} \rho_j^{-1} \int d^2\alpha\, |\psi_j\rangle_{k\cdot k}\langle\psi_j| = 1. \tag{14}$$

The proof of (14) is

$$\begin{aligned}
&\sum_{j=0}^{k-1} \rho_j^{-1} \int d^2\alpha\, |\psi_j\rangle_{k\cdot k}\langle\psi_j| \\
&= \sum_{j=0}^{k-1} \rho_j^{-1} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{\sqrt{[km+j]![kn+j]!}\, f(km+j)!\, f(kn+j)!} \\
&\quad \times \int d^2\alpha\, \frac{\alpha^{km+j}\alpha^{*(kn+j)}}{A_j(|\alpha|^2)} |km+j\rangle\langle kn+j| \\
&= \sum_{j=0}^{k-1} \rho_j^{-1} \sum_{n=0}^{\infty} 2\pi \int r\, dr\, \frac{(r^2)^{kn+j}}{A_j(r^2)[kn+j]!(f(kn+j)!)^2} |kn+j\rangle\langle kn+j| \\
&= \sum_{j=0}^{k-1} \rho_j^{-1} \sum_{n=0}^{\infty} P(kn+j)|kn+j\rangle\langle kn+j| \\
&= \sum_{j=0}^{k-1}\sum_{m=0}^{\infty} P^{-1}(km+j)|km+j\rangle\langle km+j| \times \sum_{n=0}^{\infty} P(kn+j)|kn+j\rangle\langle kn+j| \\
&= \sum_{n=0}^{\infty} |n\rangle\langle n| = 1,
\end{aligned} \tag{15}$$

where $\alpha = r\ \exp(i\theta)$ and $d^2\alpha = r\, dr\, d\theta$. Therefore, the linear combination of the k states may form a complete representation, i.e., they can be used as a representation. For example, in this representation, the q–f-coherent states (10) may be expressed as

$$|\alpha, f\rangle_q = \left[\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{[n]!(f(n)!)^2}\right]^{-1/2} \sum_{j=0}^{k-1} A_j^{1/2}(|\alpha|^2)|\psi_j\rangle_k. \tag{16}$$

3. HIGHER ORDER SQUEEZING OF THE ORTHONORMALIZED EIGENSTATES OF $(a_q f(N_q))^k$

With the development of techniques for making higher order correlation measurements in quantum optics, it is natural to turn attention to the higher order squeezing and antibunching effects of the radiation field. In 1990, Zhang *et al.* defined the higher order squeezing of a single mode of radiation field (Zhang *et al.*, 1990b). In analogy, we introduce a higher order squeezing in terms of the quantum-analogue quadrature Hermite operators $W_1(M)$ and $W_2(M)$:

$$W_1(M) = (A^{+^M} + A^M)/2, \qquad W_2(M) = i(A^{+^M} - A^M)/2. \tag{17}$$

It can be proved that the operators $W_1(M)$ and $W_2(M)$ satisfy the commutation relation

$$[W_1(M), W_2(M)] = (i/2)[A^M, A^{+^M}], \tag{18}$$

and the uncertainty relation

$$\langle(\Delta W_1)^2\rangle \cdot \langle(\Delta W_2)^2\rangle \geq \frac{1}{16}|\langle[A^M, A^{+^M}]\rangle|^2. \tag{19}$$

A state is said to be squeezed to order M if

$$\langle(\Delta W_i)^2\rangle - \frac{1}{4}|\langle[A^M, A^{+^M}]\rangle| < 0 \qquad (i = 1, 2). \tag{20}$$

From (17) and (20), we can see that it is the higher order squeezing defined by Zhang *et al.* (1990b) when $q \to 1$ and $f(n) \to 1$. Therefore, it is formally similar to that of the higher order squeezing defined by Zhang *et al.* (1990b). This kind of higher order squeezing is a natural generalization of the higher order squeezing defined by Zhang *et al*. We call it as the Mth-order squeezing effect.

Now we study the properties of the Mth-order squeezing for the k eigenstates given by (7) in following four cases.

3.1. When $M = km$ $(m = 1, 2, 3, \ldots)$, for Even and Odd k

In this case, for all of the states given by (7), using relations (Liu, 2000)

$$A^k|\psi_j\rangle_k = \alpha^k|\psi_j\rangle_k, \qquad {}_k\langle\psi_i|\psi_j\rangle_k = \delta_{ij}, \tag{21}$$

we have

$$ {}_k\langle\psi_j|A^{+^{2M}}|\psi_j\rangle_k = r^{2km}\,e^{-i2km\theta}, \qquad {}_k\langle\psi_j|A^{2M}|\psi_j\rangle_k = r^{2km}\,e^{i2km\theta}, \tag{22} $$

$$ {}_k\langle\psi_j|A^{+^{M}}|\psi_j\rangle_k = r^{km}\,e^{-ikm\theta}, \qquad {}_k\langle\psi_j|A^{M}|\psi_j\rangle_k = r^{km}\,e^{ikm\theta}, \tag{23} $$

$$ {}_k\langle\psi_j|A^{+^{M}}A^{M}|\psi_j\rangle_k = r^{2km}, \tag{24} $$

where $\alpha = r\exp(i\theta)$. Substituting (22)–(24) into (20), for the k states, it reads

$$ {}_k\langle\psi_j|(\Delta W_i)^2|\psi_j\rangle_k - \frac{1}{4}|{}_k\langle\psi_j|[A^M, A^{+^M}]|\psi_j\rangle_k| = 0 \qquad (i = 1, 2), \tag{25} $$

which indicates that the k eigenstates of (7) are all minimum uncertainty states of the operators $W_1(M)$ and $W_2(M)$ ($M = km$, $m = 1, 2, 3, \ldots$) defined by (17).

3.2. When $M = km + i$ ($m = 0, 1, 2, \ldots$; $i = 1, 2, \ldots, k-1$), for Odd k

Under this condition, for all k eigenstates of (7), we have

$$ {}_k\langle\psi_j|A^{+^{2M}}|\psi_j\rangle_k = {}_k\langle\psi_j|A^{2M}|\psi_j\rangle_k = {}_k\langle\psi_j|A^{+^{M}}|\psi_j\rangle_k = {}_k\langle\psi_j|A^{M}|\psi_j\rangle_k = 0. \tag{26} $$

Using relation (Liu, 2000)

$$ A^i|\psi_0|\psi_0\rangle_k = \alpha^i A_0^{-1/2} A_{k-i}^{1/2}|\psi_{k-i}\rangle_k \qquad (i = 1, 2, \ldots, k), \tag{27} $$

we obtain

$$ {}_k\langle\psi_S|A^{+^{M}}A^{M}|\psi_S\rangle_k = r^{2(km+i)}A_{k-i+s}/A_S, \qquad (S = 0, 1, 2, \ldots, i-1), \tag{28} $$

$$ {}_k\langle\psi_t|A^{+^{M}}A^{M}|\psi_t\rangle_k = r^{2(km+i)}A_{t-i}/A_t \qquad (t = i, i+1, \ldots, k-1). \tag{29} $$

Thus, for the states $|\psi_S\rangle_k$ ($S = 0, 1, 2, \ldots, i-1$) and $|\psi_t\langle_k$ ($t = i, i+1, \ldots, k-1$), we have

$$ {}_k\langle\psi_S|(\Delta W_1)^2|\psi_S\rangle_k - \frac{1}{4}|{}_k\langle\psi_S|[A^M, A^{+^M}]|\psi_S\rangle_k| = \frac{1}{2}r^{2(km+i)}A_{k-i+S}/A_S, \tag{30} $$

$$ {}_k\langle\psi_t|(\Delta W_1)^2|\psi_t\rangle_k - \frac{1}{4}|{}_k\langle\psi_t|[A^M, A^{+^M}]|\psi_t\rangle_k| = \frac{1}{2}r^{2(km+i)}A_{t-i}/A_t. \tag{31} $$

In order to check whether (30) and (31) satisfy (20), we need to study the values of A_j ($j = 0, 1, 2, \ldots, k-1$) in the region of $r = |\alpha| > 0$. As $[n] > [n-1] \geq 1$ for all values of q when $n \geq 0$ according to (8) and f is chosen to be nonnegative, from (9) it can be seen that $A_j(r^2) > 0$ ($j = 0, 1, 2, \ldots, k-1$) in the region $r = |\alpha| > 0$. Then the right-hand sides of (30) and (31) are larger than zero. Therefore (30) and (31) do not satisfy (20), in other words, none of the k eigenstates given by (7) exhibits Mth-order ($M = km + i$; $m = 0, 1, 2, \ldots$; $i = 1, 2, \ldots, k-1$) squeezing effect in these conditions.

3.3. When $M = (m + 1/2)k$ ($m = 0, 1, 2, \ldots$), for Even k

In this case, we have

$$_k\langle\psi_j|A^{+^{2M}}|\psi_j\rangle_k = r^{(2m+1)k}\, e^{-i(2m+1)k\theta}, \quad _k\langle\psi_j|A^{2M}|\psi_j\rangle_k = r^{(2m+1)k}\, e^{i(2m+1)k\theta}, \tag{32}$$

$$_k\langle\psi_j|A^{+^M}|\psi_j\rangle_k = {}_k\langle\psi_j|A^M|\psi_j\rangle_k = 0. \tag{33}$$

Making use of (27), we get

$$_k\langle\psi_S|A^{+^M} A^M|\psi_S\rangle_k = r^{(2m+1)k} A_{k/2+S}/A_S \quad (S = 0, 1, 2, \ldots, k/2 - 1), \tag{34}$$

$$_k\langle\psi_t|A^{+^M} A^M|\psi_t\rangle_k = r^{(2m+1)k} A_{t-k/2}/A_t \quad (t = k/2, k/2 + 1, \ldots, k - 1). \tag{35}$$

Consequently, for the states $|\psi_S\rangle_k$ ($S = 0, 1, \ldots, k/2 - 1$), we find

$$\begin{aligned} &_k\langle\psi_S|(\Delta W_1)^2|\psi_S\rangle_k - \frac{1}{4}|_k\langle\psi_S|[A^M, A^{+^M}]|\psi_S\rangle_k| \\ &= \frac{1}{2} r^{(2m+1)^k}\{A_{k/2+S}/A_S + \cos((2m + 1)k\theta)\}, \end{aligned} \tag{36}$$

and for the states $|\psi\rangle_k$ ($t = k/2, k/2 + 1, \ldots, k - 1$), we find

$$\begin{aligned} &_k\langle\psi_t|(\Delta W_1)^2|\psi_t\rangle_k - \frac{1}{4}|_k\langle\psi_t|[A^M, A^{+^M}]|\psi_t\rangle_k| \\ &= \frac{1}{2} r^{(2m+1)^k}\{A_{t-k/2}/A_t + \cos((2m + 1)k\theta)\}. \end{aligned} \tag{37}$$

According to (36) and (37), the conditions which ensure the existence of the Mth-order [$M = (m + 1/2)k$, $m = 0, 1, 2, \ldots$,] squeezing effect in the states $|\psi_S\rangle_k$($S = 0, 1, 2, \ldots, k/2 - 1$) and $|\psi_t\rangle_k$ ($t = k/2, k/2 + 1, \ldots, k - 1$) are, respectively,

$$A_{k/2+S}/A_S + \cos\{(2m + 1)k\theta\} < 0, \tag{38}$$

$$A_{t-k/2}/A_t + \cos\{(2m + 1)k\theta\} < 0. \tag{39}$$

Choose $\theta = \pi/\{(2m + 1)k\}$, so that $\cos\{(2m + 1)k\theta\} = -1$. From (7), we have $A_{k/2+S}/A_S < 1$ when $r = |\alpha| \leq 1$. Thus, (38) holds for $r \leq 1$. For $k \geq 3$, in the regions of $r > 1$, there surely exist such values of r that $A_{t-k/2}/A_t < 1$. Therefore, (39) holds. In summary, there exists Mth-order [$M = (m + 1/2)k; m = 0, 1, 2, \ldots$] squeezing effect among the k eigenstates given by (7) for even k.

3.4. When $M = km + i$ ($m = 0, 1, \ldots; i = 1, 2, \ldots, k/2 - 1, k/2 + 1, \ldots, k - 1$), for Even k

With the above discussion, it can be proved that in this case none of the k eigenstates given by (7) has the Mth-order squeezing effect.

4. QUANTUM ANALOGUE OF ANTIBUNCHING FOR THE k EIGENSTATES OF THE OPERATOR $(a_q f(N_q))^k$

In analogy with the definition of antibunching (Walls, 1983) for photon statistic properties of the radiation field, we introduce the second-order quantum-correlation function for the states given by (7) as

$$g_j^{(2)}(0) = \frac{{}_k\langle\psi_j|A^{+^2}A^2|\psi_j\rangle_k}{{}_k\langle\psi_j|A^+A|\psi_j\rangle_k^2} \quad (j = 0, 1, 2, \ldots, k-1). \tag{40}$$

The states $|\psi_j\rangle_k$ are said to have antibunching effect if $g_j^{(2)}(0) < 1$. From (40), we can see that it is the antibunching effect of a light field (Walls, 1983) when $q \to 1$ and $f(n) \to 1$. Therefore, this kind of antibunching effect is a natural generalization of the antibunching effect of a light field. It is formally similar to that of the antibunching effect of a light field (Walls, 1983).

Now we study the antibunching effect for the k eigenstates given by (7).

Using (27) and (40), for the k states given by (7), we obtain

$$g_0^{(2)}(0) = \frac{{}_k\langle\psi_0|A^{+^2}A^2|\psi_0\rangle_k}{{}_k\langle\psi_0|A^+A|\psi_0\rangle_k^2} = \frac{A_0A_{k-2}}{A_{k-1}^2}, \tag{41}$$

$$g_1^{(2)}(0) = \frac{{}_k\langle\psi_1|A^{+^2}A^2|\psi_1\rangle_k}{{}_k\langle\psi_1|A^+A|\psi_1\rangle_k^2} = \frac{A_1A_{k-1}}{A_0^2}, \tag{42}$$

$$g_j^{(2)}(0) = \frac{{}_k\langle\psi_2|A^{+^2}A^2|\psi_2\rangle_k}{{}_k\langle\psi_j|A^+A|\psi_j\rangle_k^2} = \frac{A_{j-2}A_j}{A_{j-1}^2} \quad (j = 2, 3, \ldots, k-1). \tag{43}$$

Evidently, the following relation exists $\prod_{j=0}^{k-1} g_j^{(2)}(0) = 1$.

Substituting (9) into (41), it follows that

$$g_0^{(2)}(0) = \frac{\sum_{m=0}^{\infty}\left\{\sum_{n=0}^{m}\frac{(f(kn)!)^{-2}(f(km-kn+k-2)!)^{-2}}{[kn]![km-kn+k-2]!}\right\}x^{km}}{x^k\sum_{m=0}^{\infty}\left\{\sum_{n=0}^{m}\frac{(f(kn+k-1)!)^{-2}(f(km-kn+k-1)!)^{-2}}{[kn+k-1]![km-kn+k-1]!}\right\}x^{km}}$$
$$= \varphi_1(x)/\{x^k\varphi_2(x)\}, \tag{44}$$

where $x = r^2 = |\alpha|^2$. When $f(i) \leq f(i+1)$, for $k \geq 3$, we have

$$\sum_{n=0}^{m}\frac{1}{[kn]!(f(kn)!)^2[km-kn+k-2]!(f(km-kn+k-2)!)^2}$$
$$> \sum_{n=0}^{m}\frac{1}{[kn+k-1]!(f(kn+k-1)!)^2[km-kn+k-1]!(f(km-kn+k-1)!)^2} \tag{45}$$

and thus $\varphi_1(x) > \varphi_2(x)$ for $x > 0$ when $f(i) \leq f(i+1)$. Hence $g_0^{(2)}(0) > 1$ when

$x \le 1$. However, when $x > 1$ and $f(i) \le f(i+1)$, there surely exist values of x [e.g., $x^k > \varphi_1(x)/\varphi_2(x)$] for which the following relation holds:

$$g_0^{(2)}(0) = \varphi_1(x)/\{x^k \varphi_2(x)\} < 1. \tag{46}$$

Substituting (9) into (42), we have

$$g_1^{(2)}(0) = \frac{x^k \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{m} \frac{(f(kn+1)!)^{-2}(f(km-kn+k-1)!)^{-2}}{[kn+1]![km-kn+k-1]!} \right\} x^{km}}{\sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{m} \frac{(f(kn)!)^{-2}(f(km-kn)!)^{-2}}{[kn]![km-kn]!} \right\} x^{km}}$$

$$= x^k \varphi_3(x)/\varphi_4(x). \tag{47}$$

Obviously, when $f(i) \le f(i+1)$, we have

$$\sum_{n=0}^{m} \frac{1}{[kn+1]!(f(kn+1)!)^2[km-kn+k-1]!(f(km-kn+k-1)!)^2}$$
$$< \sum_{n=0}^{m} \frac{1}{[kn]!(f(kn)!)^2[km-kn]!(f(km-kn)!)^2}, \tag{48}$$

so that $\varphi_3(x) < \varphi_4(x)$. Therefore $g_1^{(2)}(0) < 1$ when $x^k < \varphi_4(x)/\varphi_3(x)$.

From (9) and (43), we obtain

$$g_j^{(2)}(0) = \frac{\sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{m} \frac{(f(kn+j-2)!)^{-2}(f(km-kn+j)!)^{-2}}{[kn+j-2]![km-kn+j]!} \right\} x^{km}}{\sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{m} \frac{(f(kn+j-1)!)^{-2}(f(km-kn+j-1)!)^{-2}}{[kn+j-1]![km-kn+j-1]!} \right\} x^{km}}$$
$$(j = 2, 3, \ldots, k-1). \tag{49a}$$

When $f(i) \le f(i+1)$, we have

$$g_j^{(0)}(0) < \frac{\sum_{m=0}^{\infty} \frac{m+1}{[j-2]!(f(j-2)!)^2[j]!(f(j)!)^2} x^{km}}{\sum_{m=0}^{\infty} \frac{m+1}{\{[km+j-1]!(f(km+j-1)!)^2\}^2}}$$
$$< \frac{\frac{1}{[j]!(f(j)!)^2[j-2]!(f(j-2)!)^2} \sum_{m=0}^{\infty}[m+1]x^{km}}{\{[j-1]!(f(j-1)!)^2\}^{-2}}$$
$$(j = 2, 3, \ldots, k-1). \tag{49b}$$

Obviously,

$$\lim_{x\to 0}\sum_{m=0}^{\infty}[m+1]x^{km} = [1] = 1. \tag{50}$$

Therefore, from (49a), (49b), and (50), we obtain

$$\begin{aligned}\lim_{x\to 0} g_j^{(2)}(0) &< \frac{\{[j-1]!(f(j-1)!)^2\}^2}{[j]!(f(j)!)^2[j-2]!(f(j-2)!)^2}\\ &= \frac{[j-1]f^2(j-1)}{[j]f^2(j)} \qquad (j = 2, 3, \ldots, k-1).\end{aligned} \tag{51}$$

It can be seen that there is the antibunching effect in the states $|\psi_j\rangle_k$ $(j = 2, 3, \ldots, k-1)$ when $x \to 0$ and $f(j-1) \leq f(j)$.

We sum up the above results and obtain that in some different ranges of $x = |\alpha|^2$, there is the antibunching effect in all of the k states given by (7) when $f(i) \leq f(i+1)$.

5. CONCLUSIONS

In this paper, the completeness of the k orthonormalized eigenstates of the operator $(a_q f(N_q))^k (k \geq 3)$ is proved. We defined a new kind of higher order squeezing and antibunching effect, and studied the nonclassical properties of the k eigenstates, including higher order squeezing and antibunching effect. from the above discussions, for the k eigenstates of the operator $(a_q f(N_q))^k$, we come to the following conclusions:

(a) Their linear combination may form a complete representation.
(b) For odd k, none of them has the higher order squeezing effect.
(c) For odd and even k, all of them are the minimum uncertainty states of the operators $W_1(M)$ and $W_2(M)$ $(M = km, m = 1, 2, 3, \ldots)$ defined by (17).
(d) For even k, when $M = (m + 1/2)k$ $(m = 0, 1, 2, \ldots)$, all of them exhibit the Mth-order squeezing effect.
(e) There is the antibunching effect in all of the k eigenstates.

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